

# REGULAR PSEUDO-HYPEROVALS AND REGULAR PSEUDO-OVALS IN EVEN CHARACTERISTIC

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**ABSTRACT.** S. Rottey and G. Van de Voorde characterized regular pseudo-ovals of  $\mathbf{PG}(3n-1, q)$ ,  $q = 2^h$ ,  $h > 1$  and  $n$  prime. Here an alternative proof is given and slightly stronger results are obtained.

## 1. INTRODUCTION

Pseudo-ovals and pseudo-hyperovals were introduced in [10]; see also [12]. These objects play a key role in the theory of translation generalized quadrangles [6, 12]. Pseudo-hyperovals only exist in even characteristic. A characterization of regular pseudo-ovals in odd characteristic was given in [2]; see also [12]. In [8] a characterization of regular pseudo-ovals and regular pseudo-hyperovals in  $\mathbf{PG}(3n-1, q)$ ,  $q$  even,  $q \neq 2$  and  $n$  prime, is obtained. Here a shorter proof is given and slightly stronger results are obtained.

## 2. OVALS AND HYPEROVALS

A  $k$ -arc in  $\mathbf{PG}(2, q)$  is a set of  $k$  points,  $k \geq 3$ , no three of which are collinear. Any non-singular conic of  $\mathbf{PG}(2, q)$  is a  $(q+1)$ -arc. If  $\mathcal{K}$  is any  $k$ -arc of  $\mathbf{PG}(2, q)$ , then  $k \leq q+2$ . For  $q$  odd  $k \leq q+1$  and for  $q$  even a  $(q+1)$ -arc extends to a  $(q+2)$ -arc; see [3]. A  $(q+1)$ -arc is an *oval*; a  $(q+2)$ -arc,  $q$  even, is a *complete oval* or *hyperoval*.

A famous theorem of B. Segre [9] tells us that for  $q$  odd every oval of  $\mathbf{PG}(2, q)$  is a non-singular conic. For  $q$  even, there are many ovals that are not conics [3]; also, there are many hyperovals that do not contain a conic [3].

## 3. GENERALIZED OVALS AND HYPEROVALS

Arcs, ovals and hyperovals can be generalized by replacing their points with  $m$ -dimensional subspaces to obtain generalized  $k$ -arcs, generalized ovals and generalized hyperovals. These have strong connections to generalized quadrangles, projective planes, circle geometries, flocks and other structures. See [6, 12, 10, 11, 2, 7]. Below, some basic definitions and results are formulated; for an extensive study, many applications and open problems, see [12].

A *generalized  $k$ -arc* of  $\mathbf{PG}(3n-1, q)$ ,  $n \geq 1$ , is a set of  $k$   $(n-1)$ -dimensional subspaces of  $\mathbf{PG}(3n-1, q)$  every three of which generate  $\mathbf{PG}(3n-1, q)$ . If  $q$  is odd then  $k \leq q^n + 1$ , if  $q$  is even then  $k \leq q^n + 2$ . Every generalized  $(q^n + 1)$ -arc of  $\mathbf{PG}(3n-1, q)$ ,  $q$  even, can be extended to a generalized  $(q^n + 2)$ -arc.

If  $\mathcal{O}$  is a generalized  $(q^n + 1)$ -arc in  $\mathbf{PG}(3n - 1, q)$ , then it is a *pseudo-oval* or *generalized oval* or  $[n - 1]$ -*oval* of  $\mathbf{PG}(3n - 1, q)$ . For  $n = 1$ , a  $[0]$ -oval is just an oval of  $\mathbf{PG}(2, q)$ . If  $\mathcal{O}$  is a generalized  $(q^n + 2)$ -arc in  $\mathbf{PG}(3n - 1, q)$ ,  $q$  even, then it is a *pseudo-hyperoval* or *generalized hyperoval* or  $[n - 1]$ -*hyperoval* of  $\mathbf{PG}(3n - 1, q)$ . For  $n = 1$ , a  $[0]$ -hyperoval is just a hyperoval of  $\mathbf{PG}(2, q)$ .

If  $\mathcal{O} = \{\pi_0, \pi_1, \dots, \pi_{q^n}\}$  is a pseudo-oval of  $\mathbf{PG}(3n - 1, q)$ , then  $\pi_i$  is contained in exactly one  $(2n - 1)$ -dimensional subspace  $\tau_i$  of  $\mathbf{PG}(3n - 1, q)$  which has no point in common with  $(\pi_0 \cup \pi_1 \cup \dots \cup \pi_{q^n}) \setminus \pi_i$ , with  $i = 0, 1, \dots, q^n$ ; the space  $\tau_i$  is the *tangent space* of  $\mathcal{O}$  at  $\pi_i$ . For  $q$  even the  $q^n + 1$  tangent spaces of  $\mathcal{O}$  contain a common  $(n - 1)$ -dimensional space  $\pi_{q^n+1}$ , the *nucleus* of  $\mathcal{O}$ ; also,  $\mathcal{O} \cup \{\pi_{q^n+1}\}$  is a pseudo-hyperoval of  $\mathbf{PG}(3n - 1, q)$ . For  $q$  odd, the tangent spaces of a pseudo-oval  $\mathcal{O}$  are the elements of a pseudo-oval  $\mathcal{O}^*$  in the dual space of  $\mathbf{PG}(3n - 1, q)$ .

#### 4. REGULAR PSEUDO-OVALS AND PSEUDO-HYPEROVALS

In the extension  $\mathbf{PG}(3n - 1, q^n)$  of  $\mathbf{PG}(3n - 1, q)$ , consider  $n$  subplanes  $\xi_i$ ,  $i = 1, 2, \dots, n$ , that are conjugate in the extension  $\mathbb{F}_{q^n}$  of  $\mathbb{F}_q$  and which span  $\mathbf{PG}(3n - 1, q^n)$ . This means that they form an orbit of the Galois group corresponding to this extension and span  $\mathbf{PG}(3n - 1, q^n)$ .

In  $\xi_1$  consider an oval  $\mathcal{O}_1 = \{x_0^{(1)}, x_1^{(1)}, \dots, x_{q^n}^{(1)}\}$ . Further, let  $x_i^{(1)}, x_i^{(2)}, \dots, x_i^{(n)}$ , with  $i = 0, 1, \dots, q^n$ , be conjugate in  $\mathbb{F}_{q^n}$  over  $\mathbb{F}_q$ . The points  $x_i^{(1)}, x_i^{(2)}, \dots, x_i^{(n)}$  define an  $(n - 1)$ -dimensional subspace  $\pi_i$  over  $\mathbb{F}_q$  for  $i = 0, 1, \dots, q^n$ . Then,  $\mathcal{O} = \{\pi_0, \pi_1, \dots, \pi_{q^n}\}$  is a generalized oval of  $\mathbf{PG}(3n - 1, q)$ . These objects are the *regular* or *elementary pseudo-ovals*. If  $\mathcal{O}_1$  is replaced by a hyperoval, and so  $q$  is even, then the corresponding  $\mathcal{O}$  is a *regular* or *elementary pseudo-hyperoval*.

All known pseudo-ovals and pseudo-hyperovals are regular.

#### 5. CHARACTERIZATIONS

Let  $\mathcal{O} = \{\pi_0, \pi_1, \dots, \pi_{q^n}\}$  be a pseudo-oval in  $\mathbf{PG}(3n - 1, q)$ . The tangent space of  $\mathcal{O}$  at  $\pi_i$  will be denoted by  $\tau_i$ , with  $i = 0, 1, \dots, q^n$ . Choose  $\pi_i$ ,  $i \in \{0, 1, \dots, q^n\}$ , and let  $\mathbf{PG}(2n - 1, q) \subseteq \mathbf{PG}(3n - 1, q)$  be skew to  $\pi_i$ . Further, let  $\tau_i \cap \mathbf{PG}(2n - 1, q) = \eta_i$  and  $\langle \pi_i, \pi_j \rangle \cap \mathbf{PG}(2n - 1, q) = \eta_j$ , with  $j \neq i$ . Then  $\{\eta_0, \eta_1, \dots, \eta_{q^n}\} = \Delta_i$  is an  $(n - 1)$ -spread of  $\mathbf{PG}(2n - 1, q)$ .

Now, let  $q$  be even and let  $\pi$  be the nucleus of  $\mathcal{O}$ . Let  $\mathbf{PG}(2n - 1, q) \subseteq \mathbf{PG}(3n - 1, q)$  be skew to  $\pi$ . If  $\zeta_j = \mathbf{PG}(2n - 1, q) \cap \langle \pi, \pi_j \rangle$ , then  $\{\zeta_0, \zeta_1, \dots, \zeta_{q^n}\} = \Delta$  is an  $(n - 1)$ -spread of  $\mathbf{PG}(2n - 1, q)$ .

Next, let  $q$  be odd. Choose  $\tau_i$ , with  $i \in \{0, 1, \dots, q^n\}$ . If  $\tau_i \cap \tau_j = \delta_j$ , with  $j \neq i$ , then  $\{\delta_0, \delta_1, \dots, \delta_{i-1}, \pi_i, \delta_{i+1}, \dots, \delta_{q^n}\} = \Delta_i^*$  is an  $(n - 1)$ -spread of  $\tau_i$ .

Finally, let  $q$  be even and let  $\mathcal{O} = \{\pi_0, \pi_1, \dots, \pi_{q^n+1}\}$  be a pseudo-hyperoval in  $\mathbf{PG}(3n - 1, q)$ . Choose  $\pi_i$ , with  $i \in \{0, 1, \dots, q^n + 1\}$ , and let  $\mathbf{PG}(2n - 1, q) \subseteq \mathbf{PG}(3n - 1, q)$  be skew to  $\pi_i$ . Let  $\langle \pi_i, \pi_j \rangle \cap \mathbf{PG}(2n - 1, q) = \eta_j$ , with  $j \neq i$ . Then  $\{\eta_0, \eta_1, \dots, \eta_{i-1}, \eta_{i+1}, \dots, \eta_{q^n+1}\} = \Delta_i$  is an  $(n - 1)$ -spread of  $\mathbf{PG}(2n - 1, q)$ .

**Theorem 5.1** (Casse, Thas and Wild [2]). *Consider a pseudo-oval  $\mathcal{O}$  with  $q$  odd. Then at least one of the  $(n - 1)$ -spreads  $\Delta_0, \Delta_1, \dots, \Delta_{q^n}, \Delta_0^*, \Delta_1^*, \dots, \Delta_{q^n}^*$  is regular*

if and only if they all are regular if and only if the pseudo-oval  $\mathcal{O}$  is regular. In such a case  $\mathcal{O}$  is essentially a conic over  $\mathbb{F}_{q^n}$ .

**Theorem 5.2** (Rottey and Van de Voorde [8]). *Consider a pseudo-oval  $\mathcal{O}$  in  $\mathbf{PG}(3n-1, q)$  with  $q = 2^h$ ,  $h > 1$ ,  $n$  prime. Then  $\mathcal{O}$  is regular if and only if all  $(n-1)$ -spreads  $\Delta_0, \Delta_1, \dots, \Delta_{q^n}$  are regular.*

## 6. ALTERNATIVE PROOF AND IMPROVEMENTS

**Theorem 6.1.** *Consider a pseudo-hyperoval  $\mathcal{O}$  in  $\mathbf{PG}(3n-1, q)$ ,  $q = 2^h$ ,  $h > 1$  and  $n$  prime. Then  $\mathcal{O}$  is regular if and only if all  $(n-1)$ -spreads  $\Delta_i$ , with  $i = 0, 1, \dots, q^n+1$ , are regular.*

*Proof.* If  $\mathcal{O}$  is regular, then clearly all  $(n-1)$ -spreads  $\Delta_i$ , with  $i = 0, 1, \dots, q^n+1$ , are regular.

Conversely, assume that the  $(n-1)$ -spreads  $\Delta_0, \Delta_1, \dots, \Delta_{q^n+1}$  are regular. Let  $\mathcal{O} = \{\pi_0, \pi_1, \dots, \pi_{q^n+1}\}$  and let  $\hat{\mathcal{O}} = \{\beta_0, \beta_1, \dots, \beta_{q^n+1}\}$  be the dual of  $\mathcal{O}$ , with  $\beta_i$  being the dual of  $\pi_i$ .

Choose  $\beta_i, i \in \{0, 1, \dots, q^n+1\}$ , and let  $\beta_i \cap \beta_j = \alpha_{ij}, j \neq i$ . Then

$$(1) \quad \{\alpha_{i0}, \alpha_{i1}, \dots, \alpha_{i,i-1}, \alpha_{i,i+1}, \dots, \alpha_{i,q^n+1}\} = \Gamma_i$$

is an  $(n-1)$ -spread of  $\beta_i$ .

Now consider  $\beta_i, \beta_j, \Gamma_i, \Gamma_j, \alpha_{ij}, j \neq i$ . In  $\Gamma_j$  we next consider a  $(n-1)$ -regulus  $\gamma_j$  containing  $\alpha_{ij}$ . The  $(n-1)$ -regulus  $\gamma_j$  is a set of maximal spaces of a Segre variety  $\mathcal{S}_{1;n-1}$ ; see Section 4.5 in [4]. The  $(n-1)$ -regulus  $\gamma_j$  and the  $(n-1)$ -spread  $\Gamma_i$  of  $\beta_i$  generate a regular  $(n-1)$ -spread  $\Sigma(\gamma_j, \Gamma_i)$  of  $\mathbf{PG}(3n-1, q)$ . This can be seen as follows. The elements of  $\Gamma_i$  intersect  $n$  lines  $U_1, U_2, \dots, U_n$  which are conjugate in  $\mathbb{F}_{q^n}$  over  $\mathbb{F}_q$ , that is, they form an orbit of the Galois group corresponding to this extension. Let  $\alpha_{ij} \cap U_l = \{u_l\}$ , with  $l = 1, 2, \dots, n$ . Now consider the transversals  $T_1, T_2, \dots, T_n$  of the elements of  $\gamma_j$ , with  $T_l$  containing  $u_l$ . The  $n$  planes  $T_l U_l = \theta_l$  intersect all elements of  $\gamma_j$  and  $\Gamma_i$ . The  $(n-1)$ -dimensional subspaces of  $\mathbf{PG}(3n-1, q)$  intersecting  $\theta_1, \theta_2, \dots, \theta_n$  are the elements of the regular  $(n-1)$ -spread  $\Sigma(\gamma_j, \Gamma_i)$ . The elements of this spread are the points of a plane  $\mathbf{PG}(2, q^n)$ , with as lines the  $(2n-1)$ -dimensional spaces containing at least two (and then  $q^n+1$ ) elements of the spread. Hence the  $q+2$  elements of  $\hat{\mathcal{O}}$  containing an element of  $\gamma_j$ , say  $\beta_i = \beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_{q+1}}, \beta_{i_{q+2}} = \beta_j$ , are lines of  $\mathbf{PG}(2, q^n)$ . Dualizing, the elements  $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_{q+2}}$  are points of  $\mathbf{PG}(2, q^n)$ .

Now consider  $\beta_{i_2}$  and  $\gamma_j$ , and repeat the argument. Then there arise  $n$  planes  $\theta'_l$  intersecting all elements of  $\gamma_j$  and  $\Gamma_{i_2}$ . The  $(n-1)$ -dimensional subspaces of  $\mathbf{PG}(3n-1, q)$  intersecting  $\theta'_1, \theta'_2, \dots, \theta'_n$  are the elements of the regular  $(n-1)$ -spread  $\Sigma(\gamma_j, \Gamma_{i_2})$ . The elements of this spread are the points of a plane  $\mathbf{PG}'(2, q^n)$ , with as lines the  $(2n-1)$ -dimensional spaces containing  $q^n+1$  elements of the spread. Hence  $\beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_{q+2}}$  are lines of  $\mathbf{PG}'(2, q^n)$ . Dualizing, the elements  $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_{q+2}}$  are points of  $\mathbf{PG}'(2, q^n)$ .

First, assume that  $\{\theta_1, \theta_2, \dots, \theta_n\} \cap \{\theta'_1, \theta'_2, \dots, \theta'_n\} = \emptyset$ . Consider  $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$ . The planes of  $\mathbf{PG}(3n-1, q^n)$  intersecting these four spaces constitute a set  $\mathcal{M}$  of maximal spaces of a Segre variety  $\mathcal{S}_{2;n-1}$  [1]. The planes  $\theta_1, \theta_2, \dots, \theta_n, \theta'_1, \theta'_2, \dots, \theta'_n$  are elements of  $\mathcal{M}$ . It follows that  $(\theta_1 \cup \theta_2 \cup \dots \cup \theta_n) \cap (\theta'_1 \cup \theta'_2 \cup \dots \cup \theta'_n) = \emptyset$ .

Consider any  $(n-1)$ -dimensional subspace  $\pi \in \{\pi_{i_5}, \pi_{i_6}, \dots, \pi_{i_{q+2}}\}$  of  $\mathbf{PG}(3n-1, q)$ . We will show that  $\pi$  is a maximal subspace of  $\mathcal{S}_{2;n-1}$ . Let  $\theta_i \cap \pi_j = \{t_{ij}\}$ ,  $\theta'_i \cap \pi_j = \{t'_{ij}\}$ ,  $i = 1, 2, \dots, n$ ,  $j = i_1, i_2, \dots, i_{q+2}$ . If  $t_{ij_1} t_{ij_2} \cap t_{ij_3} t_{ij_4} = \{v_i\}$ ,  $t'_{ij_1} t'_{ij_2} \cap t'_{ij_3} t'_{ij_4} = \{v'_i\}$ , with  $j_1, j_2, j_3, j_4$  distinct, then  $v_1, v_2, \dots, v_n$  are conjugate and similarly  $v'_1, v'_2, \dots, v'_n$  are conjugate. Hence  $\langle v_1, v_2, \dots, v_n \rangle = \langle v'_1, v'_2, \dots, v'_n \rangle$  defines a  $(n-1)$ -dimensional space over  $\mathbb{F}_q$  which intersects  $\theta_1, \theta_2, \dots, \theta'_n$  (over  $\mathbb{F}_{q^n}$ ). The points  $t_{ij}$ , with  $j = i_1, i_2, \dots, i_{q+2}$ , generate a subplane of  $\theta_i$ , and the points  $t'_{ij}$ , with  $j = i_1, i_2, \dots, i_{q+2}$ , generate a subplane of  $\theta'_i$ , with  $i = 1, 2, \dots, n$ . Let  $q = 2^h$  and let  $\mathbb{F}_{2^v}$  be the subfield of  $\mathbb{F}_{q^n} = \mathbb{F}_{2^{hn}}$  over which these subplanes are defined; so  $v|hn$ . Then  $v < hn$  as otherwise the spreads of  $\mathbf{PG}(3n-1, q)$  defined by  $\theta_1, \theta_2, \dots, \theta_n$  and  $\theta'_1, \theta'_2, \dots, \theta'_n$  coincide, clearly not possible. The  $(n-1)$ -regulus  $\gamma_j$  implies that the subplanes contain a line over  $\mathbb{F}_q$ , so  $h|v$ . As  $n$  is prime we have  $v = h$ , so  $2^v = q$ . Hence the  $2n$  subplanes are defined over  $\mathbb{F}_q$ . It follows that the  $q+2$  elements  $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_{q+2}}$  are maximal subspaces of the Segre variety  $\mathcal{S}_{2;n-1}$ . Hence  $\pi$  is a maximal subspace of  $\mathcal{S}_{2;n-1}$ . It follows that  $\pi_1, \pi_2, \dots, \pi_{q+2}$  are maximal subspaces of  $\mathcal{S}_{2;n-1}$ .

Now consider a  $\mathbf{PG}(2, q)$  which intersects  $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$ . The  $(n-1)$ -dimensional spaces  $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_{q+2}}$  are maximal spaces of  $\mathcal{S}_{2;n-1}$  which intersect  $\mathbf{PG}(2, q)$ ; they are maximal spaces of the Segre variety  $\mathcal{S}_{2;n-1} \cap \mathbf{PG}(3n-1, q)$  of  $\mathbf{PG}(3n-1, q)$ .

Consider  $\pi_{i_1}$  and also a  $\mathbf{PG}(2n-1, q)$  skew to  $\pi_{i_1}$ . If we project  $\pi_{i_2}, \pi_{i_3}, \dots, \pi_{i_{q+2}}$  from  $\pi_{i_1}$  onto  $\mathbf{PG}(2n-1, q)$ , then by the foregoing paragraph the  $q+1$  projections constitute a  $(n-1)$ -regulus of  $\mathbf{PG}(2n-1, q)$ . Similarly, if we project from  $\pi_{i_s}$ ,  $s$  any element of  $\{1, 2, \dots, q+2\}$ . Equivalently, if  $s \in \{1, 2, \dots, q+2\}$  then the spaces  $\beta_{i_s} \cap \beta_{i_t}$ , with  $t = 1, 2, \dots, s-1, s+1, \dots, q+2$ , form a  $(n-1)$ -regulus of  $\beta_{i_s}$ .

Now assume that the condition  $\{\theta_1, \theta_2, \dots, \theta_n\} \cap \{\theta'_1, \theta'_2, \dots, \theta'_n\} = \emptyset$  is satisfied for any choice of  $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$ . In such a case every  $(n-1)$ -regulus contained in a spread  $\Gamma_s$  defines a Segre variety  $\mathcal{S}_{2;n-1}$  over  $\mathbb{F}_q$ . Let us define the following design  $\mathcal{D}$ . Points of  $\mathcal{D}$  are the elements of  $\hat{\mathcal{O}}$ , a block of  $\mathcal{D}$  is a set of  $q+2$  elements of  $\hat{\mathcal{O}}$ , containing at least one space of a  $(n-1)$ -regulus contained in some regular spread  $\Gamma_s$ , and incidence is containment. Then  $\mathcal{D}$  is a  $4-(q^n+2, q+2, 1)$  design. By Kantor [5] this implies that  $q = 2$ , a contradiction.

Consequently, we may assume that for at least one quadruple  $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$  we have

$$(2) \quad \{\theta_1, \theta_2, \dots, \theta_n\} = \{\theta'_1, \theta'_2, \dots, \theta'_n\}.$$

In such a case the  $q^n+2$  elements of  $\hat{\mathcal{O}}$  are lines of the plane  $\mathbf{PG}(2, q^n)$ . It follows that  $\mathcal{O}$  is regular.  $\blacksquare$

**Theorem 6.2.** Consider a pseudo-oval  $\mathcal{O}$  in  $\mathbf{PG}(3n-1, q)$ , with  $q = 2^h$ ,  $h > 1$  and  $n$  prime. Then  $\mathcal{O}$  is regular if and only if all  $(n-1)$ -spreads  $\Delta_0, \Delta_1, \dots, \Delta_{q^n}$  are regular.

*Proof.* If  $\mathcal{O}$  is regular, then clearly all  $(n-1)$ -spreads  $\Delta_0, \Delta_1, \dots, \Delta_{q^n}$  are regular.

Conversely, assume that the  $(n-1)$ -spreads  $\Delta_0, \Delta_1, \dots, \Delta_{q^n}$  are regular. Let  $\mathcal{O} = \{\pi_0, \pi_1, \dots, \pi_{q^n}\}$ , let  $\pi_{q^n+1}$  be the nucleus of  $\mathcal{O}$ , let  $\bar{\mathcal{O}} = \mathcal{O} \cup \{\pi_{q^n+1}\}$ , let  $\hat{\mathcal{O}}$  be the dual of  $\mathcal{O}$ , let  $\bar{\hat{\mathcal{O}}}$  be the dual of  $\bar{\mathcal{O}}$ , and let  $\beta_i$  be the dual of  $\pi_i$ .

Choose  $\beta_i, i \in \{0, 1, \dots, q^n+1\}$ , and let  $\beta_i \cap \beta_j = \alpha_{ij}, j \neq i$ . Then

$$(3) \quad \{\alpha_{i0}, \alpha_{i1}, \dots, \alpha_{i,i-1}, \alpha_{i,i+1}, \dots, \alpha_{i,q^n+1}\} = \Gamma_i$$

is an  $(n-1)$ -spread of  $\beta_i$ .

Now consider  $\beta_i, \beta_j, \Gamma_i, \Gamma_j, \alpha_{ij}$ , with  $j \neq i$  and  $i, j \in \{0, 1, \dots, q^n\}$ . In  $\Gamma_j$  we next consider a  $(n-1)$ -regulus  $\gamma_j$  containing  $\alpha_{ij}$  and  $\alpha_{j, q^n+1}$ . The  $(n-1)$ -regulus  $\gamma_j$  is a set of maximal spaces of a Segre variety  $\mathcal{S}_{1; n-1}$ . The  $(n-1)$ -regulus  $\gamma_j$  and the  $(n-1)$ -spread  $\Gamma_i$  of  $\beta_i$  generate a regular  $(n-1)$ -spread  $\Sigma(\gamma_j, \Gamma_i)$  of  $\mathbf{PG}(3n-1, q)$ . Such as in the proof of Theorem 6.1 we introduce the elements  $U_l, u_l, T_l, \theta_l, l = 1, 2, \dots, n$ , and the plane  $\mathbf{PG}(2, q^n)$ . The  $q+2$  elements of  $\hat{\mathcal{O}}$  containing an element of  $\gamma_j$ , say  $\beta_i = \beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_q}, \beta_j = \beta_{i_{q+1}}, \beta_{q^n+1}$ , are lines of  $\mathbf{PG}(2, q^n)$ . Dualizing, the elements  $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_{q+1}}, \pi_{q^n+1}$  are points of  $\mathbf{PG}(2, q^n)$ .

Now consider  $\beta_{i_2}$  and  $\gamma_j$ , and repeat the argument. Then there arise  $n$  planes  $\theta'_l$  of  $\mathbf{PG}(3n-1, q^n)$  intersecting all elements of  $\gamma_j$  and  $\Gamma_{i_2}$ , and a  $(n-1)$ -spread  $\Sigma(\gamma_j, \Gamma_{i_2})$  of  $\mathbf{PG}(3n-1, q)$ . The elements of this spread are the points of a plane  $\mathbf{PG}'(2, q^n)$ . The spaces  $\beta_{i_1}, \beta_{i_2}, \dots, \beta_{i_{q+1}}, \beta_{q^n+1}$  are lines of  $\mathbf{PG}'(2, q^n)$ . Dualizing, the elements  $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_{q+1}}, \pi_{q^n+1}$  are points of  $\mathbf{PG}'(2, q^n)$ .

First, assume that  $\{\theta_1, \theta_2, \dots, \theta_n\} \cap \{\theta'_1, \theta'_2, \dots, \theta'_n\} = \emptyset$ . Consider  $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$ . The planes of  $\mathbf{PG}(3n-1, q^n)$  intersecting these four spaces constitute a set  $\mathcal{M}$  of maximal spaces of a Segre variety  $\mathcal{S}_{2; n-1}$ . The planes  $\theta_1, \theta_2, \dots, \theta_n, \theta'_1, \theta'_2, \dots, \theta'_n$  are elements of  $\mathcal{M}$ . It follows that  $(\theta_1 \cup \theta_2 \cup \dots \cup \theta_n) \cap (\theta'_1 \cup \theta'_2 \cup \dots \cup \theta'_n) = \emptyset$ . Let  $\pi \in \{\pi_{i_5}, \pi_{i_6}, \dots, \pi_{i_{q+1}}, \pi_{q^n+1}\}$ . As in the proof of Theorem 6.1 one shows that  $\pi$  is a maximal subspace of  $\mathcal{S}_{2; n-1}$ . It follows that  $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_{q+1}}, \pi_{q^n+1}$  are maximal subspaces of  $\mathcal{S}_{2; n-1}$ .

Next consider a  $\mathbf{PG}(2, q)$  which intersects  $\pi_{i_1}, \pi_{i_2}, \pi_{i_3}, \pi_{i_4}$ . The  $(n-1)$ -dimensional spaces  $\pi_{i_1}, \pi_{i_2}, \dots, \pi_{i_{q+1}}, \pi_{q^n+1}$  are maximal spaces of  $\mathcal{S}_{2; n-1}$  which intersect the plane  $\mathbf{PG}(2, q)$ ; they are maximal spaces of the Segre variety  $\mathcal{S}_{2; n-1} \cap \mathbf{PG}(3n-1, q)$  of  $\mathbf{PG}(3n-1, q)$ . Such as in the proof of Theorem 6.1 it follows that the spaces  $\beta_{q^n+1} \cap \beta_{i_t}$ , with  $t = 1, 2, \dots, q+1$ , form a  $(n-1)$ -regulus of  $\beta_{q^n+1}$ .

Now assume that the condition  $\{\theta_1, \theta_2, \dots, \theta_n\} \cap \{\theta'_1, \theta'_2, \dots, \theta'_n\} = \emptyset$  is satisfied for any choice of  $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$ ,  $j \neq i$  and  $i, j \in \{0, 1, \dots, q^n\}$ . Let  $\alpha_1, \alpha_2, \alpha_3$  be distinct elements of  $\Gamma_{q^n+1}$ . Then  $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$  can be chosen in such a way that  $\alpha_1 \in \beta_i, \alpha_2 \in \beta_j, \alpha_2 \in \gamma_j, \beta_{i_2} \cap \beta_j \in \gamma_j$  with  $\alpha_3 \in \beta_{i_2}$ . Hence the  $(n-1)$ -regulus in  $\beta_{q^n+1}$  defined by  $\alpha_1, \alpha_2, \alpha_3$  is subset of  $\Gamma_{q^n+1}$ . From [4] now follows that the  $(n-1)$ -spread  $\Gamma_{q^n+1}$  of  $\beta_{q^n+1}$  is regular. By Theorem 6.1 the pseudo-hyperoval  $\bar{\mathcal{O}}$  is regular, and so  $\mathcal{O}$  is regular. But in such a case the condition  $\{\theta_1, \theta_2, \dots, \theta_n\} \cap \{\theta'_1, \theta'_2, \dots, \theta'_n\} = \emptyset$  is never satisfied, a contradiction.

Consequently, we may assume that for at least one quadruple  $\beta_i, \beta_j, \gamma_j, \beta_{i_2}$  we have  $\{\theta_1, \theta_2, \dots, \theta_n\} = \{\theta'_1, \theta'_2, \dots, \theta'_n\}$ . In such a case the  $q^n+2$  elements of  $\bar{\mathcal{O}}$  are lines of the plane  $\mathbf{PG}(2, q^n)$ . It follows that  $\bar{\mathcal{O}}$ , and hence also  $\mathcal{O}$ , is regular. ■

**Theorem 6.3.** Consider a pseudo-hyperoval  $\mathcal{O}$  in  $\mathbf{PG}(3n-1, q)$ ,  $q = 2^h, h > 1$  and  $n$  prime. Then  $\mathcal{O}$  is regular if and only if at least  $q^n - 1$  elements of  $\{\Delta_0, \Delta_1, \dots, \Delta_{q^n+1}\}$  are regular.

*Proof.* If  $\mathcal{O}$  is regular, then clearly all  $(n-1)$ -spreads  $\Delta_i$ , with  $i = 0, 1, \dots, q^n+1$ , are regular.

Conversely, assume that  $\rho$ , with  $\rho \geq q^n - 1$ , elements of  $\{\Delta_0, \Delta_1, \dots, \Delta_{q^n+1}\}$  are regular.

If  $\rho = q^n + 2$ , then  $\mathcal{O}$  is regular by Theorem 6.1; if  $\rho = q^n + 1$ , then  $\mathcal{O}$  is regular by Theorem 6.2.

Now assume that  $\rho = q^n$  and that  $\Delta_2, \Delta_3, \dots, \Delta_{q^n+1}$  are regular. We have to prove that  $\Delta_0$  is regular. We use the arguments in the proof of Theorem 6.2. If one of the elements  $\alpha_1, \alpha_2, \alpha_3$ , say  $\alpha_1$ , in the proof of Theorem 6.2 is  $\beta_0 \cap \beta_1$ , then let  $\gamma_j$  contain  $\beta_j \cap \beta_i, \beta_j \cap \beta_0, \beta_j \cap \beta_1$  and let  $\beta_{i_2} \neq \beta_1$ , with  $i, j \in \{2, 3, \dots, q^n + 1\}$ . Now see the proof of the preceding theorem.

Finally, assume that  $\rho = q^n - 1$  and that  $\Delta_3, \Delta_4, \dots, \Delta_{q^n+1}$  are regular. We have to prove that  $\Delta_0$  is regular. We use the arguments in the proof of Theorem 6.2. If exactly one of the elements  $\alpha_1, \alpha_2, \alpha_3$ , say  $\alpha_1$ , in the proof of Theorem 6.2 is  $\beta_0 \cap \beta_1$  or  $\beta_0 \cap \beta_2$ , then proceed as in the preceding paragraph with  $\beta_{i_2} \neq \beta_1, \beta_2$ . Now assume that two of the elements  $\alpha_1, \alpha_2, \alpha_3$ , say  $\alpha_1$  and  $\alpha_2$ , are  $\beta_0 \cap \beta_1$  and  $\beta_0 \cap \beta_2$ . Now consider all  $(n-1)$ -reguli in  $\Delta_0$  containing  $\alpha_1$  and  $\alpha_3$ , and assume, by way of contradiction, that no one of these  $(n-1)$ -reguli contains  $\alpha_2$ . The number of these  $(n-1)$ -reguli is  $\frac{q^n-2}{q-1}$ , and so  $q = 2$ , a contradiction. It follows that the  $(n-1)$ -regulus in  $\beta_0$  defined by  $\alpha_1, \alpha_2, \alpha_3$  is contained in  $\Delta_0$ . Now we proceed as in the proof of Theorem 6.2. ■

## 7. FINAL REMARKS

### 7.1. The cases $q = 2$ and $n$ not prime

For  $q = 2$  or  $n$  not prime other arguments have to be developed.

### 7.2. Improvement of Theorem 6.3

Let  $\mathcal{D} = (P, B, \in)$  be an incidence structure satisfying the following conditions.

- (i)  $|P| = q^n + 1$ ,  $q$  even,  $q \neq 2$ ;
- (ii) the elements of  $B$  are subsets of size  $q + 1$  of  $P$  and every three distinct elements of  $P$  are contained in at most one element of  $B$ ;
- (iii)  $Q$  is a subset of size  $\delta$  of  $P$  such that any triple of elements in  $P$  with at most one element in  $Q$ , is contained in exactly one element of  $B$ ;

**Assumption :** Any such  $\mathcal{D}$  is a  $3 - (q^n + 1, q + 1, 1)$  design whenever  $\delta \leq \delta_0$  with  $\delta_0 \leq q - 2$ .

**Theorem 7.1.** *Consider a pseudo-hyperoval  $\mathcal{O}$  in  $\mathbf{PG}(3n - 1, q)$ ,  $q = 2^h, h > 1$  and  $n$  prime. Then  $\mathcal{O}$  is regular if and only if at least  $q^n + 1 - \delta_0$  elements of  $\{\Delta_0, \Delta_1, \dots, \Delta_{q^n+1}\}$  are regular.*

*Proof.* Similar to the proof of Theorem 6.3. ■

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